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Separation of coupled systems of differential equations by Darboux transformations

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Abstract. We show that certain classes of coupled Schrödinger equations in one dimension which appear in various physical applications can be uncoupled by the application of an appropriate Darboux transformation.

1. Introduction

Coupled systems of Schrödinger equations

$$(\nabla^2 + k_i^2 + u_{ii}(r))\psi_i(r) = \sum_{j\neq i} u_{ij}(r)\psi_j(r), \qquad i, j = 1, \ldots, n \tag{1.1}$$

appear in various physical applications (Mott and Massey 1965, Adamová *et al* 1984, Lane and Liu 1964, Lane 1980). When the coupling terms in these equations are small it is relatively easy to obtain satisfactory approximate solutions for such systems but these methods are inadequate in the strong coupling case. In an attempt to resolve such problems Cao (1981, 1982) sought to decouple systems of two such equations in one dimension by a transformation of the form

$$\phi_i = \sum_{j=1}^{n} R_{ij}\psi_j. \tag{1.2}$$

It turns out however that the class of coupled systems which can be treated by such a transformation is rather limited. It is therefore our intention in this paper to show that larger classes of coupled systems can be separated by the use of a proper extension of Darboux transformations to systems of differential equations.

Historically, Darboux introduced (Darboux 1882) these transformations (which are a prototype of Lie-Backlund transformations) as an iterative procedure for the construction of potentials $u(x)$ for which all the eigenfunctions of

$$\phi'' = (u(x) + \lambda)\phi \tag{1.3}$$

(and similar equations) are explicitly known. Since then Darboux transformations were used in various mathematical (Crum 1955) and physical contexts (Miura 1976, Zheng 1984). Furthermore Chudnovsky (1978) and Chudnovsky and Chudnovsky (1979) extended these transformations to evolution equations of the form

$$\partial\psi/\partial t = L\psi \tag{1.4}$$

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where

$$L = (\partial/\partial x)^n + u_{n-2}(\partial/\partial x)^{n-2} + \dots \tag{1.5}$$

The plan of the paper is as follows.

In § 2 we review and generalise Darboux transformations to systems of coupled equations. In § 3 we find the conditions under which a system of two coupled Schrödinger equations can be separated by a Darboux transformation and finally in § 4 we consider a few examples.

2. Darboux transformations

Historically Darboux transformations were defined for a single differential equation as follows.

Definition 1. Given the equation

$$\phi'' = (u(x) + \lambda)\phi \tag{2.1}$$

we say that the transformation

$$\psi = A(x)\phi + B(x)\phi' \tag{2.2}$$

is a Darboux transformation if ψ satisfies a differential equation of the form

$$\psi'' = (v(x) + \lambda)\psi. \tag{2.3}$$

In what follows, however, we consider only the case $B = 1$.

To find under which constraints on A the transformation (2.2) represents a Darboux transformation we substitute (2.2) in (2.3) using (2.1) and equate the coefficients of ϕ, ϕ' to zero separately. We obtain

$$A'' + u' + A(u - v) = 0 \tag{2.4}$$

$$2A' + u - v = 0. \tag{2.5}$$

Eliminating $(u - v)$ between these equations and integrating yields

$$A' - A^2 + u = -\nu \tag{2.6}$$

where ν is an integration constant.

Equation (2.6) is a Ricatti equation which can be linearised by the standard transformation $A = -\zeta'/\zeta$ which leads to

$$\zeta'' = (u(x) + \nu)\zeta \tag{2.7}$$

Thus ζ is an eigenfunction of the original (2.1) with $\lambda = \nu$. From (2.5) we now infer that

$$v = u - 2(\ln \zeta)'' \tag{2.8}$$

i.e. a Darboux transformation changes the potential function $u(x)$ by $\Delta u = -2(\ln \zeta)''$ where ζ is an arbitrary eigenfunction of (2.1).

In complete analogy to the one-dimensional case we now define Darboux transformations for a system of second-order equations.

Definition 2. Let be given the system

$$\boldsymbol{\phi}'' = D(x)\boldsymbol{\phi}, \qquad \boldsymbol{\phi}(x) \in R^n \tag{2.9}$$

where

$$D(x) = \begin{pmatrix} u_1(x), \ d_{1,2}(x), \ldots, d_{1,n}(x) \\ \cdots \\ d_{n,1}(x), \ldots, d_{n,n-1}(x), \ u_n(x) \end{pmatrix} + \lambda I. \tag{2.10}$$

We say that

$$\boldsymbol{\psi} = A(x)\boldsymbol{\phi} + B(x)\boldsymbol{\phi}', \qquad A, B \in M(n) \tag{2.11}$$

is a Darboux transformation with respect to (2.9) if

$$\boldsymbol{\psi}'' = F(x)\boldsymbol{\psi} \tag{2.12}$$

where

$$F(x) = \begin{pmatrix} v_1(x), \ f_{1,2}(x), \ldots, f_{1,n}(x) \\ \cdots \\ f_{n,1}(x), \ldots, f_{n,n-1}(x), \ v_n(x) \end{pmatrix} + \lambda I. \tag{2.13}$$

Substituting (2.11) in (2.12) and using (2.9) it is easy to see that (2.11) is a Darboux transformation if and only if A, B satisfy the following system of equations:

$$A'' + 2B'D + AD + BD' = FA \tag{2.14}$$

$$2A' + BD + B'' = FB. \tag{2.15}$$

Thus for any given D, F this system represents a set of $2n^2$ linear coupled equations in $2n^2$ unknowns which can be solved analytically only in some special situations. It follows then that from a practical point of view it is necessary to simplify and reduce the number of equations in this system by introducing some ansatz on A and B. We consider some cases.

Case 1. $B = 0$. The system (2.14)-(2.15) reduces to

$$A'' + AD = FA, \qquad 2A' = 0. \tag{2.16}$$

Hence A is a constant coefficients matrix and

$$F = ADA^{-1}$$

i.e. D and F are similar matrices. This is exactly the result obtained by Cao (1981).

Case 2. $A = B$. Under this condition (2.14)-(2.15) form an overdetermined system for the entries of A and to restore the balance between the number of equations and unknowns we must consider the entries in F as unknowns. It follows then that

$$2A'(I - D) = AD' \tag{2.17}$$

and

$$F = (2A' + AD + A'')A^{-1}. \tag{2.18}$$

Using (2.17), a little algebra then yields

$$F = A\{D + D'(I-D)^{-1} + \tfrac{1}{2}[\tfrac{3}{2}D'(I-D)^{-1}D' + D''](I-D)^{-1}\}A^{-1}. \tag{2.19}$$

We infer then that A is completely determined by D and this determines the matrix F. If F is required to be of a special form e.g. diagonal (for the new equations in ψ to decouple) (2.19) is the proper constraint on D (and hence A) for this to happen.

Case 3. $B = I$. The system (2.14)-(2.15) reduces to

$$A'' + D' + AD = FA \tag{2.20}$$

$$2A' + D = F. \tag{2.21}$$

Hence

$$A'' - 2A'A + [A, D] + D' = 0 \tag{2.22}$$

where $[A, D] = AD - DA$. Note that (2.22) is non-linear in A. Moreover the same considerations as in the previous case apply to F.

Case 4. $A = I$. The derived equation for B in this case is

$$B'' - 2B'D + [B, D] = BD'B \tag{2.23}$$

furthermore

$$F = (BD)' + D + B'D. \tag{2.24}$$

3. Decoupling of systems of two equations

In this section we investigate the use of Darboux transformation with $B = 1$ to decouple systems of two second-order equations where the coupling terms are symmetric. Thus we consider systems where

$$D = \begin{pmatrix} u_1(x) + \lambda & d(x) \\ d(x) & u_2(x) + \lambda \end{pmatrix} \tag{3.1}$$

and require F to be of the form

$$F = \begin{pmatrix} v_1(x) + \lambda & 0 \\ 0 & v_2(x) + \lambda \end{pmatrix}. \tag{3.2}$$

From (2.21) we easily obtain the following equations for the entries a_{ij} of A;

$$2a'_{12} = 2a'_{21} = -d \tag{3.3}$$

$$2a'_{11} + u_1(x) = v_1(x) \tag{3.4}$$

$$2a'_{22} + u_2(x) = v_2(x). \tag{3.5}$$

Hence we deduce that

$$c = a_{12} = a_{21} = -\tfrac{1}{2} \int d(x)\, \mathrm{d}x. \tag{3.6}$$

Substituting these results in (2.20) and integrating whenever possible we obtain the following overdetermined system of four equations for a_{11}, a_{22}

$$a'_{11} - a^2_{11} = -u_1 + c^2 \tag{3.7}$$

$$a'_{22} - a^2_{22} = -u_2 + c^2 \tag{3.8}$$

$$\tfrac{1}{2}d' + a_{11}d - 2a_{22}'c = c(u_1 - u_2) \tag{3.9}$$

$$\tfrac{1}{2}d' + a_{22}d - 2a_{22}'c = c(u_2 - u_1). \tag{3.10}$$

Since (3.9), (3.10) are linear and uncoupled we can solve them to evaluate a_{11}, a_{22} in terms of $u_1 - u_2$ and d. We obtain

$$a_{11} = (1/2c)(\tfrac{1}{2}d + \alpha + I) \tag{3.11}$$

$$a_{22} = (1/2c)(\tfrac{1}{2}d - \alpha - I) \tag{3.12}$$

where α is an integration constant and

$$I = \int^x c(t)[u_2(t) - u_1(t)]\,\mathrm{d}t. \tag{3.13}$$

To derive the constraints that have to be imposed on u_1, u_2 and d for this solution to be consistent with (3.7), (3.8) we subtract (3.8) from (3.7) and use (3.11), (3.12) to obtain,

$$(a_{11} - a_{22})' - (d/2c)(a_{11} - a_{22}) = u_2 - u_1. \tag{3.14}$$

This equation however is satisfied identically by the solutions (3.11), (3.12) and therefore imposes no restrictions on d, u_1, u_2. We infer then that the only constraint that these functions have to satisfy is obtained by adding (3.7)-(3.8) which yields

$$u_1 + u_2 = 2c^2 - (d/2c)' + \tfrac{1}{2}(d/2c)^2 + (1/2c^2)(\alpha + I)^2. \tag{3.15}$$

To restate this result we observe that (3.15) implies that we can choose d, $u_1 - u_2$ as arbitrary functions and then use (3.15) to compute the corresponding $u_1 + u_2$ for which the resulting system of equations with u_1, u_2, d can be decoupled by a Darboux transformation.

Finally we also note that when a system (3.1) can be decoupled and the solutions $\boldsymbol{\psi}$ for the resulting system are known then $\boldsymbol{\phi}$ are given explicitly by

$$\boldsymbol{\phi} = [A^{-1}(A' + D) + A]^{-1}(\boldsymbol{\psi} - A^{-1}\boldsymbol{\psi}'). \tag{3.16}$$

(To derive this equation we assumed $B = 1$ and used (2.11), (2.9).)

4. Examples

The purpose of this section is to present two examples. The first of these is a straightforward application of the results derived in § 3. In the second example we examine the application of our results to a system of linearly coupled diffusion equations.

Example 1. If we choose

$$c = ax^k, \qquad u_1 - u_2 = bx^l$$

(note that the choice of c is equivalent to that of d) then using (3.15) we obtain

$$u_1 + u_2 = 2a^2x^{2k} + \frac{k(k-1)}{2x^2} + \frac{1}{2a^2x^{2k}}\left(\alpha + \frac{abx^{k+l+1}}{k+l+1}\right)^2 \tag{4.1}$$

especially if we set $\alpha = 0$ this simplifies to

$$u_1 + u_2 = 2a^2x^{2k} + \frac{k(k-1)}{2x^2} + \frac{b^2x^{2+2l}}{2(k+l+1)}. \tag{4.2}$$

The corresponding entries which decouple this system are

$$a_{11} = \frac{1}{2}\left(-\frac{k}{x} + \frac{\alpha}{ax^k} + \frac{bx^{l+1}}{2(k+l+1)}\right) \tag{4.3}$$

$$a_{22} = -\frac{1}{2}\left(\frac{k}{x} + \frac{\alpha}{ax^k} + \frac{bx^{l+1}}{(k+l+1)}\right) \tag{4.4}$$

and

$$a_{12} = a_{21} = ax^k.$$

Furthermore the potentials $v_1(x)$, $v_2(x)$ of the decoupled system are easy to compute using (3.4), (3.5).

Example 2. In a recent paper (Zulchner and Ames 1983) similarity solutions to a system of coupled diffusion equations were considered;

$$\boldsymbol{\phi}_t = \boldsymbol{\phi}_{xx} + \boldsymbol{f}(\boldsymbol{\phi}), \qquad \boldsymbol{\phi} \in R^n. \tag{4.5}$$

In this example we discuss a special case of this system, namely

$$\boldsymbol{\phi}_t = \boldsymbol{\phi}_{xx} + D(x)\boldsymbol{\phi} \tag{4.6}$$

and its decoupling by a Darboux transformation

$$\boldsymbol{\psi} = A(x)\boldsymbol{\phi} + B(x)\boldsymbol{\phi}_x \tag{4.7}$$

where $\boldsymbol{\psi}$ satisfies

$$\boldsymbol{\psi}_t = \psi_{xx} + F(x)\boldsymbol{\psi}. \tag{4.8}$$

Substituting (4.7) in (4.8) and using (4.5) we obtain

$$(A_{xx} - AD - BD_x + FA)\boldsymbol{\phi} + (2A_x + B_{xx} - BD + FB)\phi_x + 2B_x u_{xx} = 0. \tag{4.9}$$

Equating each coefficient of u, u_x, u_{xx} to zero yields:

$$B_x = 0 \tag{4.10}$$

$$A_{xx} - AD - BD_x + FA = 0 \tag{4.11}$$

$$2A_x + B_{xx} - BD + FB = 0. \tag{4.12}$$

Hence B is a matrix with constant entries and if we set $B = I$ then (4.11)-(4.12) reduce to

$$A_{xx} - AD - D_x = -FA \tag{4.13}$$

$$2A_x - D = -F. \tag{4.14}$$

However (4.13)-(4.14) are exactly the same as (2.20)-(2.21) under the formal substitution $D \to -D$, $F \to -F$. It follows then that we can decouple the system (4.2) under the same conditions that were derived earlier in § 3.

References

Adamová D, Hořejší J and Úlehla I 1984 *J. Phys. A: Math. Gen.* **17** 2621
Cao X Ch 1981 *J. Phys. A: Math. Gen.* **14** 1069
—— 1982 *J. Phys. A: Math. Gen.* **15** 2727
Chudnovsky D V 1978 *Proc. Natl. Acad. Sci.* **75** 4082
Chudnovsky D V and Chudnovsky G V 1979 *Lett. Nuovo Cimento* **25** 263
Crum M M 1955 *Q. J. Math.* **2** 121
Darboux J G 1882 *C.R. Acad. Sci. Paris* **94** 1456
Lane N F 1980 *Rev. Mod. Phys.* **52** 29
Lane N F and Liu C C 1964 *Phys. Rev.* **133** A947
Miura R (ed) 1976 *Backlund transformations* (*Lecture Notes Math.* **515**) (Berlin: Springer)
Mott N F and Massey H S 1965 *The theory of atomic collisions* (Oxford: OUP)
Zheng W M 1984 *J. Math. Phys.* **25** 88
Zulchner W and Ames W F 1983 *Nonlinear Analysis* **7** 945